Fibonacci \( Q \)– matrix and Matrices Formula for Fibonacci and Lucas Sequences

Teerapan Jodnok\(^1\) Sukanya Somprom\(^2\)

\(^1\)Department of Mathematics, Faculty of Science and Technology, Surindra Rajabhat University, Thailand
E-mail: satidkku07@gmail.com
\(^2\)Department of Mathematics, Faculty of Science and Technology, Surindra Rajabhat University, Thailand
E-mail: promsukan@hotmail.com

Abstract

In this paper, we studied and found the new matrices of \( 3 \times 3 \), which it have similar properties to Fibonacci \( Q \)– matrix. Moreover, we studied and found the matrix formula

\[
\begin{bmatrix}
0 & 2 & 0 \\
1 & 1 & 0 \\
3 & 1 & 1
\end{bmatrix}
= 
\begin{bmatrix}
F_n & L_n \\
F_{n+1} & L_{n+1} \\
F_{n+2} & L_{n+2}
\end{bmatrix}
\]

when \( F_n \) and \( L_n \) are Fibonacci and Lucas sequences, respectively.

Keywords: Fibonacci sequences, Lucas sequences, \( Q \)– matrix

1. Introduction

The Fibonacci sequences is the sequence of integer \( F_n \) defined by the initial values \( F_0 = 1, F_1 = 0 \) and the recurrence relation (Koshy, 2001).

\[
F_n = F_{n-1} + F_{n-2}
\]

for all \( n \geq 3 \).

The frist few values of \( F_n \) are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

The Lucas sequences is the sequence of integer \( L_n \) defined by the initial values \( L_0 = 2, L_1 = 1 \) and the recurrence relation (Koshy, 2001).
\[ L_n = L_{n-1} + L_{n-2} \]

for all \( n \geq 3 \).

The first few values of \( L_n \) are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...

The Fibonacci \( Q \)– matrix was first used by Brenner (Brenner, 1951), and its basic properties were enumerated by King (King, 1960).

In 1981, Gould showed that the Fibonacci \( Q \)– matrix is a square \( 2 \times 2 \) matrix of the following form,

\[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
\end{bmatrix}
\]

The following property of the \( nth \) power of \( Q \)– matrix was proved

\[
\begin{bmatrix}
F_{n+1} & F_n \\
F_n & F_{n-1} \\
\end{bmatrix}
\]  

(Gould, 1981).

In 1985, Honsberger showed that the Fibonacci \( Q \)– matrix is a square \( 2 \times 2 \) matrix of the following form,

\[
\begin{bmatrix}
F_2 & F_1 \\
F_1 & F_0 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

The following property of the \( nth \) power of \( Q \)– matrix was proved

\[
\begin{bmatrix}
F_{n+1} & F_n \\
F_n & F_{n-1} \\
\end{bmatrix}
\]  

(Honsberger, 1985).

In this paper, we studied and found the new matrices of \( 3 \times 3 \), which it have similar properties to Fibonacci \( Q \)– matrix.
2. Main Results

In this study, we studied and found the new matrices of $3 \times 3$, which have similar properties to Fibonacci $Q$ matrix. Moreover, we investigate the new property of Fibonacci and Lucas number in relation with the Fibonacci and Lucas matrices formula. We have the following theorem.

\textbf{Theorem 2.1} If 
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\] then 
\[
\begin{pmatrix}
F_{n-1} & F_{n-2} & F_n \\
F_{n-2} & F_{n-3} & F_{n-1} \\
F_{n-1} & F_{n-2} & F_n
\end{pmatrix}
\] for all integers $n \geq 3$

\textbf{Proof.} Let use the principle of mathematical induction on $n$. For $n = 3$ is true, since 
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\] 
\[
\begin{pmatrix}
0 & F_{1,1} & F_{1,2} \\
0 & F_{1,2} & F_{1,1} \\
0 & F_{1,3} & F_{1,1}
\end{pmatrix}
\] 
Assume that it is true for all positive integer $n = k$, that is
\[
\begin{pmatrix}
0 & F_{k,1} & F_{k,2} \\
0 & F_{k,2} & F_{k,1} \\
0 & F_{k,3} & F_{k,1}
\end{pmatrix}
\]
Consider for $n = k + 1$, 
\[
\begin{pmatrix}
0 & F_{k+1,1} & F_{k+1,2} \\
0 & F_{k+1,2} & F_{k+1,1} \\
0 & F_{k+1,3} & F_{k+1,1}
\end{pmatrix} = \begin{pmatrix}
0 & F_{k,1} & F_{k,1} \\
0 & F_{k,2} & F_{k,1} \\
0 & F_{k,3} & F_{k,1}
\end{pmatrix} + \begin{pmatrix}
0 & F_{1,1} & F_{1,2} \\
0 & F_{1,2} & F_{1,1} \\
0 & F_{1,3} & F_{1,1}
\end{pmatrix}
\] 
\[
= \begin{pmatrix}
0 & F_{k-2} & F_{k-1} \\
0 & F_{k-1} & F_k \\
0 & F_k & F_{k+1}
\end{pmatrix}
\] 
\[
= \begin{pmatrix}
F_{(k+1)-3} & F_{(k+1)-2} \\
F_{(k+1)-2} & F_{(k+1)-1} \\
F_{(k+1)-1} & F_{(k+1)}
\end{pmatrix}
\]
Therefore, the result is true for every \( n \geq 3 \).

**Theorem 2.2** For all \( n \in \mathbb{N} \) we have,

\[
Q = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} F_{i} & L_{i} \\ F_{i+1} & L_{i+1} \\ F_{i+2} & L_{i+2} \end{bmatrix}
\]

**Proof.** Let use the principle of mathematical induction on \( n \). For \( n = 1 \) is true, since

\[
Q^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} F_{i} & L_{i} \\ F_{i+1} & L_{i+1} \\ F_{i+2} & L_{i+2} \end{bmatrix}
\]

Assume that it is true for all positive integer \( n = k \), that is

\[
Q^k = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} F_{i} & L_{i} \\ F_{i+1} & L_{i+1} \\ F_{i+2} & L_{i+2} \end{bmatrix}
\]

Consider for \( n = k + 1 \),

\[
Q^{k+1} = (QQ^k) = Q \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}
\]

\[
= \begin{bmatrix} F_{i+1} & L_{i+1} \\ F_{i+2} & L_{i+2} \\ F_{i+2} + F_{i+1} & L_{i+2} + L_{i+1} \end{bmatrix}
\]
Therefore, the result is true for every $n \geq 1$.

Let us generalize this observation using the Fibonacci and Lucas formula matrices.

**Proposition 2.3** For all integers $m, n$ such that $3 \leq m < n$, we have the following relations

(a) $F_n = F_{m-1}F_{n-m+1} + F_{m-2}F_{n-m+2}$

(b) $L_n = F_{m-1}L_{n-m+1} + F_{m-2}L_{n-m+2}$

**Proof.** From the laws of exponent for the square matrices. So, we have

\[
Q^n = Q^n Q^n
\]

it follows that

\[
Q = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad Q^n = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}
\]

From Theorem 2.1 and Theorem 2.2, it follows that:

\[
\begin{bmatrix} F_n & L_n \\ F_{n+1} & L_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & F_{m-1} \\ F_{m-2} & F_{m} \end{bmatrix} \begin{bmatrix} F_{n-m} & L_{n-m} \\ F_{n-m+1} & L_{n-m+1} \end{bmatrix}
\]

By consider the corresponding element. That is,

\[
F_n = F_{m-1}F_{n-m+1} + F_{m-2}F_{n-m+2}
\]

\[
L_n = F_{m-1}L_{n-m+1} + F_{m-2}L_{n-m+2}
\]

Completes the proof.
3. Conclusion

In this paper, we studied and found the new matrices of $3 \times 3$, which it have similar properties to Fibonacci $Q$ – matrix. Moreover, we investigate the new property of Fibonacci and Lucas number in relation with the Fibonacci and Lucas matrices formula.

References


