BI-INTERIOR IDEALS AND INTERIOR IDEAL OF ORDERED SEMIGROUPS

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ABSTRACT

In this paper, we show the way we pass from semigroups into ordered semigroups and characterize ordered semigroups in terms of bi-interior ideals. We characterize quasi-ideal, bi-ideal and interior ideal in terms of bi-interior ideals and study the properties of bi-interior ideals of ordered semigroup, simple ordered semigroup and regular ordered semigroup.

Keywords: ordered semigroup, quasi ideal, bi-ideal, interior ideal, bi-interior ideal, bi-quasi ideal, regular ordered semigroup, bi-interior simple ordered semigroup.

1. Introduction

In 1950, ordered semigroup was studied by several researchers. The theory of different types of ideals in semigroups and in ordered semigroups was studied by several researchers. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers. Ponizovskii and Tsingelis [9] studied bi-ideals in ordered semigroups. In 2006, Lee and Lee [8] gave some characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups. In 2008, Iampan [6] studied the concept of (0-) minimal and maximal ordered quasi-ideals in ordered semigroups. Ansari, Khan and Kaushik [1] characterized the notion of (m, n) quasi-ideals in semigroups. In 2010, Khan, Khan and Hussain [10] characterized regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of intuitionistic fuzzy left (resp. right) ideals. Tang and Xie [14] characterized ordered semigroups in which the radical of every ideal (right ideal, bi-ideal) is an ordered subsemigroup (resp., ideal, right ideal, left ideal, bi-ideal, interior ideal) by using some binary relations on an ordered semigroup.

As a further generalization of ideals, Murali Krishna Rao [8] introduced the notion of bi-interior ideal of semigroup as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of semigroup. In this paper, we will present the notion of bi-interior ideal of ordered semigroup.

2. Objectives

We extend the notion of bi-interior ideals of ordered semigroup, as a generalization of bi-ideals and interior ideals of ordered semigroup and study the properties of bi-interior ideals of ordered semigroup.

3. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 3.1. [7] An ordered semigroup we mean a structure $(S, \cdot, \leq)$ in which the following are satisfied:

(i) $(S, \cdot)$ is a semigroup,

(ii) $(S, \leq)$ is a poset,

(iii) $a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb.$ for all $a, b, x \in S.$

For a non-empty subset $A \subseteq S,$ we denote $(A] := \{t \in S \mid t \leq h$ for some $h \in A\}.$ If $A = \{a\},$ then we write $(a]$ instead of $(\{a\}].$ For non-empty subsets $A, B \subseteq S,$ we denote, $(AB] := \{ab \mid a \in A, b \in B\}.$

We mention the properties we use in the paper. Clearly $S = (S],$ and for subset A, B of $S,$ we have the following:

- $A \subseteq (A];$
- If $A \subseteq B,$ then $(A] \subseteq (B];$
- $(A](B] \subseteq (AB];$
- $((A](B]] = ((A]B] = (A(B]] = (AB];$
- $(A \cap B] \subseteq (A] \cap (B]$
- $(A \cup B] = (A] \cup (B];$
- $((A]] = (A].$

Definition 3.2. [2] Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A.$

Definition 3.3. [5] Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *quasi-ideal* of S if:

(i) $(AS] \cap (SA] \subseteq A;$

(ii) For $a \in A$ and $b \in S, b \leq a \Rightarrow b \in A,$

(equivalently $(A] \subseteq A,$ which in turn is equivalent to $(A] = A).$

Definition 3.4. [1] A non-empty subset A of a n ordered semigroup $(S, \cdot, \leq)$ is called a *left* (resp. *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);

(ii) for $a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$ or $(A] = A$.

And a non-empty subset A of S is called an *ideal* of S if it is both a left ideal and a right ideal of S.

Definition 3.5. [7] Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup B of S is called a *bi-ideal* of S if it satisfies the following conditions:

(i) $BSB \subseteq B$;

(ii) For $a \in B$ and $b \in S, b \leq a \Rightarrow b \in B$ or $(B] = B$.

Definition 3.6. [6] Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup B of S is called an *interior ideal* of S if it satisfies the following conditions:

(i) $SBS \subseteq B$;

(ii) For $a \in B$ and $b \in S, b \leq a \Rightarrow b \in B$ or $(B] = B$.

Definition 3.7. [11] An ordered semigroup $(S, \cdot, \leq)$ is called a *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalent definitions:

(i) $A \subseteq (ASA]$ $(\forall A \subseteq S)$;

(ii) $a \in (aSa]$ $(\forall a \in S)$.

Definition 3.8. [8] An element $1 \in S$ is said to be *unity* if $x1 = 1x = x$ for all $a \in S$.

Definition 3.9. [4] An ordered semigroup $(S, \cdot, \leq)$ is called an idempotent ordered semigroup if $e \leq e^2$ for every $e \in S$.

Definition 3.9. A subsemigroup A of ordered semigroup $(S, \cdot, \leq)$ is called a *left* (resp. *right*) *bi-quasi ideal* of S if:

(i) $(SA] \cap (ASA] \subseteq A$ (resp. $(AS] \cap (ASA] \subseteq A$);

(ii) For $a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$ or $(A] = A$.

Definition 3.10. A subsemigroup A of ordered semigroup $(S, \cdot, \leq)$ is called a *bi-quasi ideal* of S if A is a left bi-quasi ideal and a right bi-quasi ideal of S.

Example 3.1. Let $S = \{a, b, c, d\}$ be an ordered semigroup with the following multiplication table

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $A = \{a\}$

We define the order relation "$\leq$" as follows:

$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, d)\}$

Then A is a bi-quasi ideal of ordered semigroup S.

Definition 3.11. [7]. An ordered semigroup $(S,\cdot,\leq)$ is said to be a *left* (resp. *right*) *simple ordered semigroup* if S does not contain proper left (resp. right) ideals of S.

Definition 3.12. An ordered semigroup $(S,\cdot,\leq)$ is a *bi-quasi simple ordered semigroup* if S has no proper bi-quasi ideal of S.

4. Bi-interior ideals of ordered semigroups

In this section, we study the properties of bi-interior ideals of ordered semigroups, simple ordered semigroups and regular ordered semigroups. And we apply S instead $(S,\cdot,\leq)$. Throughout this paper, S is an ordered semigroup with unity.

Definition 4.1. A non-empty subset B of ordered semigroup S is said to be *bi-interior ideal* of S if:

(i) $(SBS]\cap(BSB]\subseteq B$;

(ii) For $a\in B$ and $b\in S, b\leq a\Rightarrow b\in B$ or $(B]=B$.

Example 4.1. Let $S=\{a,b,c,d\}$ be an ordered semigroup with the following multiplication table

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $B=\{a\}$

We define the order relation "$\leq$" as follows:

$\leq := \{(a,a),(a,b),(b,b),(c,c),(d,d)\}$

Then B is a bi-quasi ideal of ordered semigroup S.

Definition 4.2. An ordered semigroup S is said to be *bi-interior simple ordered semigroup* if S has no bi-interior ideals other than S itself.

Theorem 4.1. Let S be an ordered semigroup. Then the following are hold.

(i) Every left ideal is a bi-interior ideal of S.

(ii) Every right ideal is a bi-interior ideal of S.

(iii) Every quasi ideal is a bi-interior ideal of S.

(iv) If L is a bi-interior ideal of ordered semigroup S then $LL\subseteq L$.

(v) Intersection of a right ideal and a left ideal of S is a bi-interior ideal of S.

(vi) If B is a bi-interior ideal of S, then $(BS]$ and $(SB]$ are bi-interior ideals of S.

(vii) If B is a bi-interior ideal and T is a right ideal of S, then $B\cap T$ is a bi-interior ideal of semigroup S.

Corollary 4.1. If A and B are bi-interior ideals of S, then $A \cap B$ is a bi-interior ideal of S.

Corollary 4.2. Every ideal is a bi-interior ideal of ordered semigroup S.

Theorem 4.2. Let S be a simple ordered semigroup. Every bi-interior ideal is an ideal of S.

Proof. Let S be a simple ordered semigroup and B be a bi-interior ideal of S. Then $(SBS] \cap (BSB] \subseteq B$. By considering $(SBS]S \subseteq (SBS](S] \subseteq (SBSS] \subseteq (SBS]$, and $S(SBS] \subseteq (S](SBS] \subseteq (SSBS] \subseteq (SBS]$. Clearly, if $x \in (SBS]$ and $y \in S$ such that $y \leq x$, we have $y \in (SBS]$. Hence $(SBS]$ is an ideal of S. Since S is a simple ordered semigroup, then $(SBS] = S$. We have $(SBS] \cap (BSB] \subseteq B$ and $(SBS] = S$, then $S \cap (BSB] \subseteq B$. Thus $(BSB] \subseteq B$. Since $BSB \subseteq (BSB] \subseteq B$, we have $BSB \subseteq B$. Clearly, $(B] = B$. Hence the theorem.

Theorem 4.3. Let S be an ordered semigroup. Then S is a bi-interior simple ordered semigroup if and only if $(SaS] \cap (aSa] = S$ for all $a \in S$.

Proof. Suppose S is a bi-interior simple ordered semigroup and $a \in S$. We have

$$\begin{aligned}(S((SaS] \cap (aSa])S] \cap (((SaS] \cap (aSa])S((SaS] \cap (aSa])] &\subseteq ((S](SaS](S]] \cap ((aSa](S](aSa]] \\ &\subseteq ((SaS]] \cap ((aSa]] \\ &= (SaS] \cap (aSa].\end{aligned}$$

Clearly, if $x \in (SaS] \cap (aSa]$ and $y \in S$ such that $y \leq x$, we have $y \in (SaS] \cap (aSa]$. Hence $(SaS] \cap (aSa]$ is a bi-interior ideal of S. Hence $(SaS] \cap (aSa] = S$, for all $a \in S$.

Conversely suppose that $(SaS] \cap (aSa] = S$, for all $a \in S$. Let B be a bi-interior ideal of ordered semigroup S and $a \in B$. Consider $S = (SaS] \cap (aSa] \subseteq (SBS] \cap (BSB] \subseteq B$. Therefore $S = B$. Hence S is a bi-interior simple ordered semigroup.

Theorem 4.4. Every bi-ideal of ordered semigroup S is a bi-interior ideal of S.

Proof. Let B be a bi-ideal of ordered semigroup S. Then $(SBS] \cap (BSB] \subseteq (BSB] \subseteq (B] = B$. Clearly, $(B] = B$. Hence every bi-ideal of ordered semigroup S is a bi-interior ideal of S.

Theorem 4.5. Every interior ideal of ordered semigroup S is a bi-interior ideal of S.

Proof. Let B be an interior ideal of ordered semigroup S. Then $(SBS] \cap (BSB] \subseteq (SBS] \subseteq (B] = B$. Clearly, $(B] = B$. Hence every interior ideal of ordered semigroup S is a bi-interior ideal of S.

Theorem 4.6. Let S be a regular ordered semigroup. Then every interior ideal of S is an ideal of S.

Proof. Let B be an interior ideal of S. Then $SBS \subseteq B$. By Theorem 3.5, we have B is a bi-interior ideal of ordered semigroup S. Let $x \in BS$. Since S is regular, we have $x \in (xSx]$ for all $x \in S$. Hence $x \leq xyx$ for some $y \in S$. By considering $x \leq xyx \in BSSBS \subseteq SBS$. Then $x \in (SBS]$. And $x \leq xyx \in BSSBS \subseteq BSB$. Hence $x \in (BSB]$. Therefore $BS \subseteq (SBS] \cap (BSB] \subseteq B$. Let $x \in SB$. Since S is regular, we have $x \in (xSx]$ for all $x \in S$. Hence $x \leq xyx$ for some $y \in S$. By considering $x \leq xyx \in SBSSB \subseteq SBS$. Then $x \in (SBS]$. And $x \leq xyx \in SBSSB \subseteq BSB$. Then $x \in (BSB]$. Therefore $SB \subseteq (SBS] \cap (BSB] \subseteq B$. Clearly, $(B] = B$. Hence B is an ideal of regular ordered semigroup S.

Theorem 4.7. If L is a minimal left ideal and R is a minimal right ideal of ordered semigroup S, then $B = (RL]$ is a minimal bi-interior ideal of S.

Proof. Let $B = (RL]$ where L is a minimal left ideal and R is a minimal right ideal of ordered semigroup S. So we have

$$\begin{aligned}(S(RL]S] \cap ((RL]S(RL]] &\subseteq ((S](RL](S]] \cap ((RL](S](RL]]\\ &\subseteq ((S(RL)S]] \cap (((RL)S(RL)]]\\ &\subseteq (S(RL)S] \cap (RL]\\ &\subseteq (RL].\end{aligned}$$

If $x \in (RL]$ and $y \in S$ such that $y \leq x$, we have $y \in (RL]$. Hence $B = (RL]$ is a bi-interior ideal of S. Suppose A is a bi-interior ideal of S, such that $A \subseteq B$. Consider $(SA] \subseteq (SB] = (S(RL]] \subseteq ((S](RL]] \subseteq ((SRL]] \subseteq ((L]] = (L] = L$, since L is a left ideal of S. Similarly, we can prove $(AS] \subseteq (BS] = ((RL]S] \subseteq ((RL](S]] \subseteq ((RLS]] \subseteq ((R]] = (R] = R$. Therefore $(SA] = L$, $(AS] = R$. Hence $B = (RL] = ((AS](SA]] \subseteq (ASSA] \subseteq (ASA]$ and $B = (RL] = ((AS](SA]] \subseteq (ASSA] \subseteq (SA] \subseteq (SAS]$. Therefore $B \subseteq (ASA] \cap (SAS] \subseteq A$. Thus $A = B$. Hence B is a minimal bi-interior ideal of S.

Theorem 4.8. Let A and C be subsemigroups of ordered semigroup S and $B = (AC]$. If A is the left ideal of S then B is a bi-interior ideal of S.

Proof. Let A and C be subsemigroups of ordered semigroup S and $B = (AC]$. Suppose A is the left ideal of S. We have

$(SBS] = ((AC]S(AC]] \subseteq ((AC](S](AC]] \subseteq ((ACSAC]] \subseteq ((ACAC]] \subseteq ((AC]] = (AC] = B$. Therefore $(BSB] \cap (SBS] \subseteq (SBS] \subseteq B$. Let $x \in B$ and $y \in S$ such that $y \leq x$, then $y \in B$. Hence B is a bi-interior ideal of S.

Corollary 4.9. Let A and C be subsemigroup of ordered semigroup S and $B = (CA]$. If C is the right ideal then B is a bi-interior ideal of S.

Proof. Let A and C be subsemigroups of ordered semigroup S and $B = (CA]$. Suppose C is the right ideal of S. We have $(SBS] = ((CA]S(CA]] \subseteq ((CA](S](CA]] \subseteq ((CASCA]] \subseteq ((CSCA]] \subseteq ((CA]] = (CA] = B$. Therefore $(BSB] \cap (SBS] \subseteq (SBS] \subseteq B$. Clearly, if $x \in B$ and $y \in S$ such that $y \leq x$, then $y \in B$. Hence B is a bi-interior ideal of S.

Lemma 4.1. [10] Let S an ordered semigroup. Then S is regular if and only if for every right ideal R and every left ideal L of S, we have $(RL] = (R \cap L]$.

Theorem 4.10. S is a regular ordered semigroup if and only if $B \cap I \cap L \subseteq BIL$, for any bi-interior ideal B, ideal I and left ideal L of S.

Proof. Suppose S is a regular ordered semigroup, B, I and L are bi-interior ideal, ideal and left ideal of respectively. Let $a \in B \cap I \cap L$. Then $a \in (aSa]$, since S is a regular. Consider $a \in (aSa] \subseteq ((aSa]S(aSa]] \subseteq (aSaSaSa] \subseteq (BSB]$, and $a \in (aSa] \subseteq ((aSa]S(aSa]] \subseteq (aSaSaSa] \subseteq (SBS]$. Hence $a \in (BSB] \cap (SBS] \subseteq B$. Therefore $B \cap I \cap L \subseteq B$.

Conversely suppose that $B \cap I \cap L \subseteq BIL$, for any bi-interior ideal B, ideal I and left ideal L of S. Let R be a right ideal and L a left ideal of S. Then by assumption, $R \cap L = R \cap S \cap L \subseteq RSL \subseteq RL \subseteq (RL]$. We have $(RL] \subseteq R$, $(RL] \subseteq L$. Therefore $(RL] \subseteq R \cap L$. Hence $R \cap L = (RL]$. Thus S is a regular ordered semigroup.

Corollary 4.11. Let S be an ordered semigroup and e be an idempotent. Then $(eS]$ and $(Se]$ are bi-interior ideal of S respectively.

Proof. Let S be an ordered semigroup and e be an idempotent. We will show that $(eS]$ and $(Se]$ are bi-interior ideal of S respectively. Consider $(eS](eS] \subseteq (eSeS] \subseteq (eS]$ and

$(S(eS]S] \cap ((eS]S(eS]] \subseteq (eSSeS] \subseteq (eS]$.

If $x \in (eS]$ and $y \in S$ such that $y \le x$, we have $y \in (eS]$. Therefore $(eS]$ is a bi-interior ideal of S. We have $(Se](Se] \subseteq (SeSe] \subseteq (Se]$ and $(S(Se]S] \cap ((Se]S(Se]] \subseteq (SeSSe] \subseteq (Se]$. If $x \in (Se]$ and $y \in S$ such that $y \le x$, we have $y \in (Se]$. Therefore $(Se]$ is a bi-interior ideal of S. Hence the theorem.

Theorem 4.12. Let B be a bi-ideal of ordered semigroup S and I be an interior ideal of S. Then $(B \cap I]$ is a bi-interior ideal of S.

Proof. Suppose B is a bi-ideal of ordered semigroup S and I is an interior ideal of S. By considering $(B \cap I](B \cap I] \subseteq ((B \cap I](B \cap I]] \subseteq (BB] \subseteq (B] = B$ and $(B \cap I](B \cap I] \subseteq ((B \cap I](B \cap I]] \subseteq (II] \subseteq (I] = I$. Then $(B \cap I](B \cap I] \subseteq B \cap I \subseteq (B \cap I]$. Hence $(B \cap I]$ is a subsemigroup of S. Therefore $(B \cap I]S(B \cap I] \subseteq (B]S(B] = BSB \subseteq B$ and $S(B \cap I]S \subseteq S(I]S = SIS \subseteq I$. Then $((B \cap I]S(B \cap I]] \cap (S(B \cap I]S] \subseteq (B] \cap (I] \subseteq B \cap I \subseteq (B \cap I]$. Clearly, if $x \in (B \cap I]$ and $y \in S$ such that $y \le x$, then $y \in (B \cap I]$. Hence $(B \cap I]$ is a bi-interior ideal of S.

Theorem 4.13. Let S be an ordered semigroup. If $S = Sa$, for all $a \in S$. Then every bi-interior ideal of S is a quasi-ideal of S.

Proof. Let B be a bi- interior ideal of ordered semigroup S and $a \in B$. Then $(SBS] \cap (BSB] \subseteq B$. By considering $Sa \subseteq SB$, then $S \subseteq SB \subseteq S$. Hence $SB = S$. By considering $(SB] \cap (BS] = (SBS] \cap (BSB] \subseteq B$. Clearly, $(B] = B$. Therefore B is a quasi-ideal of S. Hence the theorem.

Theorem 4.14. Let S be a regular ordered semigroup. Then B is a bi-interior ideal of S if and only if $(SBS] \cap (BSB] = B$, for all bi-interior ideals B of S.

Proof. Suppose S is a regular ordered semigroup, B is a bi-interior ideal of S and $x \in B$. Then $(SBS] \cap (BSB] \subseteq B$. Since x is regular, there exists $y \in S$ such that $x \le xyx \in BSB$ and $x \le xyxyx \in SBS$. Therefore $x \in (SBS] \cap (BSB]$. Hence $(SBS] \cap (BSB] = B$.

Conversely suppose that $(SBS] \cap (BSB] = BIL$, for all bi- interior ideals B of S. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of S. Therefore

$$(S(R \cap L)S] \cap ((R \cap L]S(R \cap L]] \subseteq ((R \cap L]S(R \cap L]] \subseteq (RSL] \subseteq (RL] \subseteq R \cap L$$

since $(RL] \subseteq L$ and $(RL] \subseteq R$. Hence B is a bi-interior ideal of S.

Theorem 4.15. The intersection of $\{B_i \mid i \in A\}$ bi-interior ideal of an ordered semigroup S is a bi-interior ideal of S.

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a subsemigroup of S. Since B_λ is a bi-interior ideal of S, we have $(B_\lambda S B_\lambda] \cap (S B_\lambda S] \subseteq B_\lambda$, for all $\lambda \in A$. By considering $((\bigcap B_\lambda) S (\bigcap B_\lambda)] \cap (S(\bigcap B_\lambda)S] \subseteq (B_\lambda S B_\lambda] \cap (S B_\lambda S]$, for all $\lambda \in A$. Hence $((\bigcap_{\lambda \in A} B_\lambda) S (\bigcap_{\lambda \in A} B_\lambda)] \cap (S(\bigcap_{\lambda \in A} B_\lambda)S] \subseteq (\bigcap_{\lambda \in A} B_\lambda)$. Therefore $(BSB] \cap (SBS] \subseteq B$. Let $x \in B$ and $y \in S$ such that $y \leq x$, then $y \in B$. Hence B is a bi-interior ideal of S.

Theorem 4.16. Let B be subsemigroup of a regular ordered semigroup S. Then B can be represented as $B = (RL]$, where R is a right ideal and L is a left ideal of S if and only if B is a bi-interior ideal of B.

Proof. Suppose $B = (RL]$, where R is a right ideal and L is a left ideal of regular ordered semigroup S. We have $(BSB] = ((RL]S(RL]] \subseteq ((RLSRL]] \subseteq (RL]$, and $(SBS] \cap (BSB] \subseteq (BSB] \subseteq (RL] = B$. Clearly, $((R]] = RL$. Hence B is a bi-interior ideal of S.

Conversely suppose that B is a bi-interior ideal of regular ordered semigroup S. By Theorem 4.14, we have $(SBS] \cap (BSB] = B$. Let $R = (BS]$ and $L = (SB]$. Consider $(BS)S \subseteq (BSS) \subseteq (BS] = R$. Clearly, $((BS]] = (BS]$, Then $(BS] = (R]$ Then is a right ideal of S. $S(SB) \subseteq (SSB) \subseteq (SSB) \subseteq (SB] = L$. Clearly, $((SB]] = SB$, Then $(SB] = L$ is a left ideal of S. We have $(BS] \cap (SB] \subseteq (SBS] \cap (BSB] = B$, then $(BS] \cap (SB] \subseteq B$, thus $R \cap L \subseteq B$. We have $B \subseteq (BS] = R$ and $B \subseteq (SB] = L$. Then $B \subseteq R \cap L$, thus $B = R \cap L = (RL]$, since S is a regular semigroup. Hence B can be represented as $(RL]$, where R is the right ideal and L is the left ideal of S. Hence the theorem.

Theorem 4.17. If B is an interior ideal of regular ordered semigroup S, then B is a bi-quasi ideal of S.

Proof. Suppose B is an interior ideal of regular ordered semigroup S. By Theorem 4.5, we have B is a bi-interior ideal of S. Let $x \in (SB]$. Hence $x \in (xSx]$. Then $x \leq xsx$ for some $y \in S$. By considering $x \leq xyx \in SBSSB \subseteq SBS$. Then $x \in (SBS]$. Let $x \in (BS]$. Therefore $x \in (xSx]$. Then $x \leq xsx$ for some $y \in S$. Consider $x \leq xyx \in BSSBS \subseteq SBS$. Hence $x \in (SBS]$. Then $(SB] \cap (BSB] \subseteq (SBS] \cap (BSB] \subseteq B$ and $(BS] \cap (BSB] \subseteq (SBS] \cap (BSB] \subseteq B$. Hence B is a bi-quasi ideal of S.

Theorem 3.18. If ordered semigroup S is a left (right) simple ordered semigroup then every bi-interior ideal of S. is a right (left) ideal of S.

Proof. Let B be a bi-interior ideal of left simple ordered semigroup S. Consider $(SB](SB] \subseteq (SB]$. Then $S(SB] \subseteq (SSB] \subseteq (SB]$. If $x \in (SB]$ and $y \in S$ such that $y \leq x$, then $y \in (SB]$. And $(SB] \subseteq S$. Therefore $(SB]$ is a left ideal of S. Hence $(SB] = S$. Then $(BS] \subseteq (SS] = ((SB]S] \subseteq ((SBS]] \subseteq (SBS]$ and $(BS] \subseteq (B(BS]] \subseteq (BSB]$. Hence $BS \subseteq (BS] \subseteq (SBS] \cap (BSB] \subseteq B$. Therefore B is a right ideal of S. Let B be a bi-interior ideal of right simple ordered semigroup S. Consider $(BS](BS] \subseteq (BS]$. Then $(BS]S \subseteq (BSS] \subseteq (BS]$. If $x \in (BS]$ and $y \in S$ such that $y \leq x$, then $y \in (BS]$. And $(BS] \subseteq S$. Therefore $(BS]$ is a right ideal of S. Hence $(B] = S$. Then

$(SB] \subseteq (SS] = ((BS]B] \subseteq ((BSB]] = (BSB]$ and $(SB] \subseteq (S(SB]] \subseteq (SBS]$. Hence $SB \subseteq (SB] \subseteq (BSB] \cap (SBS] \subseteq B$. Therefore B is a left ideal of S.

Corollary 4.3. If ordered semigroup S is a simple ordered semigroup then bi-interior ideal of S is an ideal of S.

5. Conclusion

Here we introduced the notion of bi-interior ideal of ordered semigroup as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of ordered semigroup and studied some of their properties. We introduced the notion of bi-interior simple ordered semigroup and characterized the bi-interior simple ordered semigroup, the bi-interior regular ordered semigroup using bi-interior ideals, we proved every bi-interior ideal of ordered semigroup is a bi-quasi ideal and studied some of the properties of bi-interior ideals of ordered semigroup.

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