Fibonacci $Q-$matrix and Matrices Formula for Fibonacci and Lucas Sequences

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Abstract

In this paper, we studied and found the new matrices of 3×3, which it have similar properties to Fibonacci $Q-$matrix. Moreover, we studied and found the matrix formula

$$Q^n\begin{bmatrix}0 & 2\\1 & 1\\1 & 3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 1 & 1\end{bmatrix}^n\begin{bmatrix}0 & 2\\1 & 1\\1 & 3\end{bmatrix}=\begin{bmatrix}F_n & L_n\\F_{n+1} & L_{n+1}\\F_{n+2} & L_{n+2}\end{bmatrix}$$

when F_n and L_n are Fibonacci and Lucas sequences, respectively.

Keywords: Fibonacci sequences, Lucas sequences, $Q-$matrix

1. Introduction

The Fibonacci sequences is the sequence of interger F_n defined by the initial values $F_0=1$, $F_1=1$ and the recurrence relation (Koshy, 2001).

$$F_n=F_{n-1}+F_{n-2}$$

for all $n\geq 3$.

The frist few values of F_n are $1,1,2,3,5,8,13,21,34,55,89,144,...$

The Lucas sequences is the sequence of interger L_n defined by the initial values $L_0=2$, $L_1=1$ and the recurrence relation (Koshy, 2001).

$$L_n = L_{n-1} + L_{n-2}$$

for all $n \geq 3$.

The frist few values of L_n are $2,1,3,4,7,11,18,29,47,76,123,199,...$

The Fibonacci $Q-$matrix was first used by Brenner (Brenner, 1951), and its basic properties were enumerated by King (King, 1960).

In 1981, Gould showed that the Fibonacci $Q-$matrix is a square 2×2 matrix of the following form,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The following property of the nth power of $Q-$matrix was proved

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

(Gould, 1981).

In 1985, Honsberger showed that the Fibonacci $Q-$matrix is a square 2×2 matrix of the following form,

$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The following property of the nth power of $Q-$matrix was proved

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

(Honsberger, 1985).

In this paper, we studied and found the new matrices of 3×3, which it have similar properties to Fibonacci $Q-$matrix.

2. Main Results

In this study, we studied and found the new matrices of 3×3, which it have similar properties to Fibonacci $Q-$matrix. Moreover, we investigate the new property of Fibonacci and Lucas number in relation with the Fibonacci and Lucas matrices formula. We have the following theorem.

Theorem 2.1 If $Q=\begin{bmatrix}0&1&0\\0&0&1\\0&1&1\end{bmatrix}$ then $Q^n=\begin{bmatrix}0&F_{n-3}&F_{n-2}\\0&F_{n-2}&F_{n-1}\\0&F_{n-1}&F_n\end{bmatrix}$ for all integers $n\geq 3$

Proof. Let use the principle of mathematical induction on n. For $n=3$ is true, since

$$Q^3=\begin{bmatrix}0&1&0\\0&0&1\\0&1&1\end{bmatrix}^3=\begin{bmatrix}0&1&1\\0&1&2\\0&2&3\end{bmatrix}=\begin{bmatrix}0&F_0&F_1\\0&F_1&F_2\\0&F_2&F_3\end{bmatrix}=\begin{bmatrix}0&F_{3-3}&F_{3-2}\\0&F_{3-2}&F_{3-1}\\0&F_{3-1}&F_3\end{bmatrix}$$

Assume that it is true for all positive integer $n=k$, that is

$$Q^k=\begin{bmatrix}0&F_{k-3}&F_{k-2}\\0&F_{k-2}&F_{k-1}\\0&F_{k-1}&F_k\end{bmatrix}$$

Consider for $n=k+1$,

$$Q^{k+1}=QQ^k=\begin{bmatrix}0&1&0\\0&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}0&F_{k-3}&F_{k-2}\\0&F_{k-2}&F_{k-1}\\0&F_{k-1}&F_k\end{bmatrix}$$

$$=\begin{bmatrix}0&F_{k-2}&F_{k-1}\\0&F_{k-1}&F_k\\0&F_{k-1}+F_{k-2}&F_k+F_{k-1}\end{bmatrix}$$

$$=\begin{bmatrix}0&F_{k-2}&F_{k-1}\\0&F_{k-1}&F_k\\0&F_k&F_{k+1}\end{bmatrix}=\begin{bmatrix}0&F_{(k+1)-3}&F_{(k+1)-2}\\0&F_{(k+1)-2}&F_{(k+1)-1}\\0&F_{(k+1)-1}&F_{k+1}\end{bmatrix}$$

Therefore, the result is true for every $n \geq 3$.

Theorem 2.2 For all $n \in \mathbb{N}$ we have,

$$Q^n \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} F_n & L_n \\ F_{n+1} & L_{n+1} \\ F_{n+2} & L_{n+2} \end{bmatrix}$$

Proof. Let use the principle of mathematical induction on n. For $n=1$ is true, since

$$Q^1 \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^1 \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} F_1 & L_1 \\ F_{1+1} & L_{1+1} \\ F_{1+2} & L_{1+2} \end{bmatrix}$$

Assume that it is true for all positive integer $n = k$, that is

$$Q^k \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^k \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} F_k & L_k \\ F_{k+1} & L_{k+1} \\ F_{k+2} & L_{k+2} \end{bmatrix}$$

Consider for $n = k+1$,

$$Q^{k+1} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \left(QQ^k\right) \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = Q\left(Q^k \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} F_k & L_k \\ F_{k+1} & L_{k+1} \\ F_{k+2} & L_{k+2} \end{bmatrix}$$

$$= \begin{bmatrix} F_{k+1} & L_{k+1} \\ F_{k+2} & L_{k+2} \\ F_{k+2} + F_{k+1} & L_{k+2} + L_{k+1} \end{bmatrix}$$

$$=\begin{bmatrix} F_{k+1} & L_{k+1} \\ F_{k+2} & L_{k+2} \\ F_{k+3} & L_{k+3} \end{bmatrix} = \begin{bmatrix} F_{k+1} & L_{k+1} \\ F_{(k+1)+1} & L_{(k+1)+1} \\ F_{(k+1)+2} & L_{(k+1)+2} \end{bmatrix}$$

Therefore, the result is true for every $n \geq 1$.

Let us generalize this observation using the Fibonacci and Lucas formula matrices.

Proposition 2.3 For all integers m, n such that $3 \leq m < n$, we have the following relations

(a) $F_n = F_{m-3}F_{n-m+1} + F_{m-2}F_{n-m+2}$

(b) $L_n = F_{m-3}L_{n-m+1} + F_{m-2}L_{n-m+2}$

Proof. From the laws of exponent for the square matrices. So, we have

$$Q^n = Q^m Q^{n-m}$$

it follows that

$$Q^n \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = Q^m \left(Q^{n-m} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \right)$$

From Theorem 2.1 and Theorem 2.2, it follows that :

$$\begin{bmatrix} F_n & L_n \\ F_{n+1} & L_{n+1} \\ F_{n+2} & L_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & F_{m-3} & F_{m-2} \\ 0 & F_{m-2} & F_{m-1} \\ 0 & F_{m-1} & F_m \end{bmatrix} \begin{bmatrix} F_{n-m} & L_{n-m} \\ F_{n-m+1} & L_{n-m+1} \\ F_{n-m+2} & L_{n-m+2} \end{bmatrix}$$

By consider the corresponding element. That is,

$$F_n = F_{m-3}F_{n-m+1} + F_{m-2}F_{n-m+2}$$

$$L_n = F_{m-3}L_{n-m+1} + F_{m-2}L_{n-m+2}$$

Completes the proof.

3. Conclusion

In this paper, we studied and found the new matrices of 3×3, which it have similar properties to Fibonacci $Q-$matrix. Moreover, we investigate the new property of Fibonacci and Lucas number in relation with the Fibonacci and Lucas matrices formula.

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